

Elements
of
Matrix Modeling and Computing
with
MATLAB

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Contents

List of Figures	vii
List of Tables	xi
Preface	xiii
Introduction	xv
1 Vectors in the Plane	1
1.1 Floating Point and Complex Numbers	1
1.2 Complex Valued Functions	10
1.3 Vectors in \mathbb{R}^2	19
1.4 Dot Product and Work	27
1.5 Lines and Curves in \mathbb{R}^2 and \mathbb{C}	38
2 Vectors in Space	47
2.1 Vectors and Dot Product	47
2.2 Cross and Box Products	56
2.3 Lines and Curves in \mathbb{R}^3	67
2.4 Planes in \mathbb{R}^3	76
2.5 Extensions to \mathbb{R}^n	86
3 $Ax = d$: Unique Solution	95
3.1 Matrix Models	95
3.2 Matrix Products	105
3.3 Special Cases of $Ax = d$	117
3.4 Row Operations and Gauss Elimination	127
3.5 Inverse Matrices	138
3.6 LU Factorization	149
3.7 Determinants and Cramer's Rule	159
4 $Ax = d$: Least Squares Solution	171
4.1 Curve Fitting to Data	171
4.2 Normal Equations	182

4.3	Multilinear Data Fitting	191
4.4	Parameter Identification	199
5	Ax = d: Multiple Solutions	209
5.1	Subspaces and Solutions in \mathbb{R}^3	209
5.2	Row Echelon Form	220
5.3	Nullspaces and Equilibrium Equations	230
6	Linear Initial Value Problems	243
6.1	First Order Linear	243
6.2	Second Order Linear	250
6.3	Homogeneous and Complex Solution	258
6.4	Nonhomogeneous Linear Differential Equations	264
6.5	System Form of Linear Second Order	272
7	Eigenvalues and Differential Equations	281
7.1	Solution of $\mathbf{x}' = A\mathbf{x}$ by Elimination	281
7.2	Real Eigenvalues and Eigenvectors	289
7.3	Solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$	296
8	Image Processing in Space Domain	311
8.1	Matrices and Images	311
8.2	Contrast and Histograms	321
8.3	Blurring and Sharpening	331
9	Image Processing in Frequency Domain	343
9.1	Laplace and Fourier Transforms	343
9.2	Properties of DFT	351
9.3	DFT in $\mathbb{R}^n \times \mathbb{R}^n$	361
9.4	Frequency Filters in $\mathbb{R}^n \times \mathbb{R}^n$	370
	Appendix.	381
	A Solutions to Odd Exercises	381
	Bibliography	397
	Index	399

List of Figures

1.1.1	Complex Numbers as Arrows	4
1.1.2	Norm(z^2) and Angle(z^2)	7
1.2.1	Affine, Square and Square Root of z	13
1.2.2	Solutions of $z^{12} = 1$	17
1.3.1	A Vector in the Plane	20
1.3.2	$c^2 = a^2 + b^2$	21
1.3.3	$c^2 = b^2 + a^2 - 2ab \cos(\theta)$	22
1.3.4	$\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $s\vec{b}$	23
1.4.1	Trigonometric Identity and Dot Product	31
1.4.2	Area and Dot Product	33
1.4.3	Linearly Independent Vectors	34
1.4.4	Work and a Ramp	35
1.4.5	Torque on a Wheel	35
1.4.6	Work with Independent Paths	36
1.5.1	Line Given a Point and Direction	39
1.5.2	Minimum Distance of Point to a Line	41
1.5.3	Cycloid and Wheel	43
1.5.4	Cycloid	44
1.5.5	Two-tone Signal	45
2.1.1	Point in Space	48
2.1.2	Vector in Space	49
2.1.3	Vector Addition in Space	50
2.2.1	Unit Vector Cross Products	58
2.2.2	Projected Area	58
2.2.3	Box Product and Volume	64
2.2.4	Determinant and Volume	65
2.3.1	Vector Equation and Minimum Distance	68
2.3.2	Distance between Two Lines	71
2.3.3	Helix	73
2.3.4	Projectile in Space	75
2.4.1	Normal and Point	77
2.4.2	Three Points	78

2.4.3	Linear Combination of Vectors	79
2.4.4	Minimum Distance to a Plane	81
2.5.1	Mesh of Image Matrix	92
2.5.2	Imwrite of Image Matrix	92
2.5.3	Negative Image Matrix	92
3.1.1	Box with Fixed Volume	101
3.1.2	Cost of a Box	102
3.1.3	Two-bar Truss	102
3.1.4	Two-loop Circuit	103
3.2.1	Heat Conduction in a Wire	112
3.2.2	Steady State Heat Diffusion	114
3.3.1	Temperature in Wire with Current	125
3.4.1	Six-bar Truss	134
3.5.1	Five-bar Truss	149
3.6.1	Three-loop Circuit	155
3.6.2	Potential in a Single-loop Circuit	156
3.7.1	Three-tank Mixing	167
4.1.1	Sales Data	174
4.1.2	Least Squares Function for Sales Data	176
4.1.3	Radioactive Decay	179
4.2.1	World Population Prediction	188
4.4.1	US Population and Logistic Model	204
4.4.2	Temperature Data and Curve Fit	207
5.3.1	Bar e with Four Forces	237
5.3.2	Fluid Flow in Four Cells	239
6.2.1	Mass-Spring System	251
6.3.1	Variable Damped Mass-Spring	263
6.4.1	Forced Mass-Spring	272
6.5.1	Series LRC Circuit	274
6.5.2	Tuned Circuit with Modulated Signal	280
7.3.1	Heat Diffusion in Thin Wire	308
8.1.1	Pollen Image	312
8.1.2	Enhanced Pollen Image	312
8.1.3	Aerial Photo	313
8.1.4	Enhanced Aerial Photo	313
8.1.5	Mars Rover Photo	314
8.1.6	Enhanced Mars Rover Photo	314
8.1.7	Moon	315
8.1.8	Sharper Moon Image	315
8.1.9	Plot of the Matrix C	317

8.1.10	Image of Letter C	317
8.1.11	Negative Image	318
8.1.12	Matrix NCSU	318
8.1.13	Image of NCSU	319
8.1.14	Negative Image of NCSU	319
8.1.15	Center Grain in Pollen	320
8.2.1	Histogram of Pollen Image	322
8.2.2	Histogram of Lighter Pollen Image	324
8.2.3	Lighter Pollen Image	325
8.2.4	Piecewise Linear Function	326
8.2.5	Histogram for Enhanced Pollen Image	328
8.2.6	Higher Contrast Pollen Image	328
8.2.7	Mars Rover Image Using Power $1/2$	330
8.2.8	Mars Rover Image Using Power 2	330
8.3.1	Deblurred 1D Image	334
8.3.2	Original NCSU	336
8.3.3	Blurred NCSU	336
8.3.4	Deblurred NCSU	337
8.3.5	Increased Contrast Pollen	338
8.3.6	Brighter and Sharper Pollen	339
8.3.7	Original Moon Image	340
8.3.8	Brightened and Sharpened	341
9.2.1	DFT of Sine and Cosine	354
9.2.2	Noisy Sine Function	359
9.2.3	Filtered Sine Image	360
9.3.1	2D DFT of Sine and Cosine	364
9.3.2	Noisy 2D Sine Wave	367
9.3.3	Mesh Plot of Noisy Sine Wave	368
9.3.4	DFT of Noisy Sine Wave	368
9.3.5	Low-pass Filter	369
9.3.6	Filtered DFT of Sine Wave	369
9.3.7	Filtered Sine Wave	370
9.4.1	Noisy NCSU Image	372
9.4.2	Low-pass Filtering of NCSU	373
9.4.3	Ideal Low-pass NCSU	373
9.4.4	Band-reject Filtering of NCSU	374
9.4.5	Band-reject Filtered NCSU	374
9.4.6	Light and Noisy Aerial Image	376
9.4.7	Filtering Aerial Image	376
9.4.8	Filtered Aerial Image	377
9.4.9	Micro Chip Image	378
9.4.10	Sharpening of Micro Chip Image	378
9.4.11	Sharpened Micro Chip Image	379



List of Tables

4.1.1 : Computer Sales Data	173
4.1.2 : World Population Data	174
4.1.3 : Radioactive Decay Data	178
4.3.1 : Multilinear Data	191
4.3.2 : Price Data for Three Markets	193
4.3.3 : Home Appraisal Data	195
4.3.4 : Three-tank Mixing Data	197
4.4.1 : US Population Data	202
4.4.2 : Temperature Data	205



Preface

An important objective of this book is to provide "math-on-time" for second year students of science and engineering. The student should have had one semester of calculus. The student most likely would take this matrix course concurrently with the second semester of calculus or would use this text for independent study of these important topics. This text fills in often missed topics in the first year of calculus including complex numbers and functions, matrices, algebraic systems, curve fitting, elements of linear differential equations, transform methods and some computation tools.

Chapters one and two have introductory material on complex numbers, 2D and 3D vectors and their products, which are often covered in the beginning of multivariable calculus. Here a connection is established between the geometric and algebraic approaches to these topics. This is continued into chapters three, four and five where higher order algebraic systems are solved via row operations, inverse matrices and LU factorizations. Linearly independent vectors and subspaces are used to solve over and under determined systems. Chapters six and seven describe first and second order linear differential equations and introduce eigenvalues and eigenvectors for the solution of linear systems of initial value problems. The last two chapters use transform methods to filter distorted images or signals. The discrete Fourier transform is introduced via the continuous versions of the Laplace and Fourier transforms. The discrete Fourier transform properties are derived from the Fourier matrix representation and are used to do image filtering in the frequency domain.

The first five chapters can be used as a two-credit course (28 50-minute classes). Among the nine chapters there is more than enough material for a three-credit course. This three-credit matrix course when coupled with a nine or ten-credit calculus sequence can serve as a more "diverse" alternative to the traditional twelve-credit calculus sequence. The twelve-credit calculus sequence can be adapted to this alternative by reducing the precalculus, moving some of 2D and 3D vectors and differential equations into the matrix course, and using computing tools to do the complicated computations and graphing.

Most sections have some applications, which should indicate the utility of the mathematics being studied. Seven basic applications are developed in various sections of the text and include circuits, trusses, mixing tanks, heat conduction, data modeling, motion of a mass and image filters. The applications are

developed from very simple models to more complex models. The reader can locate sections pertaining to a particular application by using the index.

MATLAB[®] is used to do some of the more complicated computations. Although the primary focus is to develop by-hand calculation skills, most sections at the end have some MATLAB calculations. The MATLAB m-files used in the text are listed in the index and are included in the book's www site: <http://www4.ncsu.edu/~white>. The approach to using computing tools includes: first, learn the math and by-hand calculations; second, use a computing tool to confirm the by-hand calculations; third, use the computing tool to do more complicated calculations and applications.

I hope this book will precipitate discussions concerning the core mathematical course work that science and engineers are required to study. Discrete models and computing have become more common, and this has increased the need for additional study of matrix computation, numerical and linear algebra. The precise topics, skills, theory and appropriate times to teach these is certainly open for discussion. The matrix algebra topics in this book are a small subset of most upper level linear algebra courses, which should be enhanced and taken by a number of students. This book attempts to make a bridge from two and three variable problems to more realistic problems with more variables, but it emphasizes skills more than theory.

I thank my colleagues who have contributed to many discussions about the content of this text. And, many thanks goes to my personal friends and Liz White who have listened to me emote during the last year.



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Introduction

One can view an $m \times n$ matrix as a table of objects with m rows and n columns. The objects are usually real or complex numbers, but they could be characters or records of information. A simple example is data for the last 12 months of car sales where there are $m = 12$ rows and $n = 2$ columns. The first column will have the month's number and the second column will have the number of cars sold in the corresponding month. By examining the data one would like to make a prediction about futures sales. This is where the modeling enters. If the graph of the sales versus months "looks" like a straight line, then the data may be modeled by a linear function of time $y = \hat{m}t + c$. The slope \hat{m} and intercept c must be chosen so that the computed sales are "close" to the car sales data. This is done by appropriate manipulations of the two column vectors and computing a solution of the resulting system of algebraic equations. Once \hat{m} and c have been found, the predicted sales for t larger than 12 can easily be calculated by evaluating the linear function. The modeling process is complicated by incorrect sales data, changing prices and other models such as a parabolic function of time.

This text examines a variety of applications, which have matrix models and often have algebraic systems that must be solved either by-hand calculations or using a computing tool. Applications to projectiles, circuits, mixing tanks, trusses, heat conduction, motion of a mass, curve fitting and image enhancement will be initially modeled in very simple ways and then revisited so as to make the model more accurate. This is typical of the modeling process where there is an application, a model, mathematical method, computations and assessment of the results. Then this cycle is repeated so as to enhance the application's model.

The first two chapters deal with problems in two and three dimensional space where the matrices have no more than three rows or columns. Here geometric insight can be used to understand the models. In Section 2.5 the extension to higher dimensions is indicated for vectors and matrices, solution to larger algebraic systems, more complicated curve fitting, time dependent problems with systems of differential equations and image modeling. Chapters three, four and five have the basic matrix methods that are required to solve systems in higher dimensions. Chapters six and seven contain time dependent models and introduce linear systems of differential equations. The last two chapters

are an introduction to image and signal processing.

Most sections have some by-hand matrix calculations in the numbered examples, some applications and some MATLAB computations, see [4] and [6]. The focus is on the by-hand calculations, and one should carefully study the numbered examples. Each numbered example usually has two exercises associated with it. There are also additional exercises, which may fill in some parts of the text, be related to applications or use MATLAB. This text is not intended to be a tutorial on MATLAB, but there are a number of short codes that may help you understand the topics being discussed. The by-hand calculations should be done, and MATLAB should be used to confirm these calculations. This will give you confidence in both your understanding of the by-hand matrix computation and the use of MATLAB. Larger dimensional problems can easily be done using MATLAB or other computer software.

The following matrices are used in Chapters 3, 4, 5 and 9, and they can be generalized to larger matrices enabling one to cross the bridge from models with few variables to many variables.

$$\begin{aligned}
 Z &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & I &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 E_{32}(-3) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} & U &= \begin{bmatrix} 1 & 7 & -10 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \\
 [A \ d] &= \begin{bmatrix} 2 & -1 & 0 & 200 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 70 \end{bmatrix} & [U \ \hat{d}] &= \begin{bmatrix} 1 & -1 & 0 & 200 \\ 0 & 3/2 & -1 & 100 \\ 0 & 0 & 4/3 & 410/3 \end{bmatrix} \\
 A &= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} & A^{-1} &= (1/10) \begin{bmatrix} 8 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 8 \end{bmatrix} \\
 LS &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} & REF &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 \mathbb{F}_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & z & z^2 & z^3 \\ 1 & z^2 & 1 & z^2 \\ 1 & z^3 & z^2 & z \end{bmatrix}
 \end{aligned}$$