Computational Linear Algebra

Courses on linear algebra and numerical analysis need each other. Often NA courses have some linear algebra topics, and LA courses mention some topics from numerical analysis/scientific computing. This text merges these two areas into one introductory undergraduate course. It assumes students have had multivariable calculus. A second goal of this text is to demonstrate the intimate relationship of linear algebra to applications/computations.

A rigorous presentation has been maintained. A third reason for writing this text is to present, in the first half of the course, the very important topic on singular value decomposition, SVD. This is done by first restricting consideration to real matrices and vector spaces. The general inner product vector spaces are considered starting in the middle of the text.

The text has a number of applications. These are to motivate the student to study the linear algebra topics. Also, the text has a number of computations. MATLAB® is used, but one could modify these codes to other programming languages. These are either to simplify some linear algebra computation, or to model a particular application.

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with Applications and MATLAB® Computations

Robert E. White

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Preface

Courses on linear algebra and numerical analysis need each other. Often NA courses have some linear algebra topics, and LA courses mention some topics from numerical analysis/scientific computing. One reason for writing this text is to merge these two areas into one introductory undergraduate course for students with multivariable calculus. A second goal is to demonstrate the intimate relationship of linear algebra to applications/computations. A third objective is to present, in the first half of the course, the very important topic of singular value decomposition, SVD.

Of course some topics have not been fully covered. Here the rigorous presentation has been maintained. This is done by first restricting consideration to real matrices and vector spaces in $\mathbb{R}^n$. The general inner product vector spaces are considered starting in the middle of the text, Chapter 10.

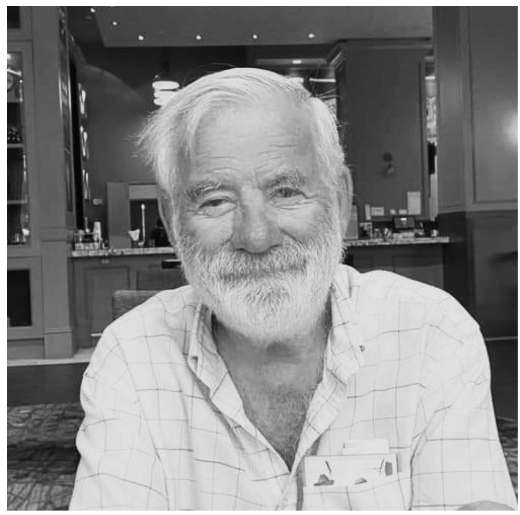
The text has a number of applications. These are to motivate the student to study the linear algebra topics. The purpose of an application is not to necessarily teach all of the application, but it is to make a case for the importance and utility of linear algebra.

Also, the text has a number of computations. Here MATLAB® [18] has been used, but one could modify these codes to other programming languages. These are either to simplify some linear algebra computation, or to model a particular application. One could invest a lot of time examining these codes, but they should be viewed as supplements and not a replacement to linear algebra topics.

Matrices with inverses are covered in the first two chapters. Least squares and under-determined systems are in Chapters 3 and 4. Orthogonality is presented in Chapters 5 and 6. Chapters 7–9 are related to the SVD. Chapter 10 covers general inner product spaces and includes topics related to boundary value problems. Basic iterative methods are introduced in Chapter 11: matrix splitting, conjugate gradient (CG) and generalized minimum residual method (GMRES). Finally, not all problems are linear, and in the last chapter introductions are given to the basic nonlinear methods: Picard, Newton and Levenberg-Marquardt.

Some topics in this text have been taken from my lecture notes and [26] [28]. These select topics are not comprehensive and neither are the remaining contents of this text. In this text I cited a number of authors and mention several here. A classic text is volume one by R. Courant and D. Hilbert, [6], and the preface and Chapters 5 and 6 are very interesting. The book by G. Golub and C. Van Loan, [9], gives a comprehensive study of matrix analysis

as it relates to numerical linear algebra. The popular book by G. Strang, [23], has a number of applications of linear algebra. The very comprehensive book by C. Meyer, [17], is a valuable resource for further study in linear algebra and applications.



Robert E White, July 28, 2022

Introduction

This text is a continuation of vector, matrices and algebraic linear systems with small dimension. Because of the technical difficulties of by-hand calculations, "small" means $n = 2, 3$ or 4. One goal is to extend this to higher dimensions and the use of computing tools. For example, fluid flow models in 3D space can have algebraic systems with very large numbers of unknowns. Consider finding air pressures inside of small cubes with 100 edges in each direction; this gives $n = 10^6$ unknowns! Linear algebra and computation can be combined to solve such problems.

The first four chapters contain extensions of some topics you have studied. Chapter 1 extends vector and matrix operations to vectors with n real numbers and to $n \times n$ matrices with n^2 components. Algebraic systems with n unknowns and n equations will be solved by Gauss elimination (row operations), inverse matrices and determinants. Four very special matrices with inverse are introduced in the second chapter. The third and fourth chapters are a study of over-determined (least squares) and under-determined (multiple solutions) algebraic systems.

Another view of the course is the study of seven factorizations of a matrix: $P^T LU$, LU, $G^T G$, QR, QDQ^T, $U\Sigma V^T$ and UTU^*. The first three will be done in Chapter 2, the next two will be presented in Chapters 5 and 6. The singular value decomposition (SVD) of an $m \times n$ matrix is $U\Sigma V^T$, and it will be studied in Chapters 7–9. The SVD has been known for well over a century, but with the use of computers it has proven useful in a number of areas such as image compression and search engines. The last factorization, UTU^*, is for matrices with complex numbers as components.

The notion of vectors is extended to functions or objects that can be added and multiplied by scalers. This is done in Chapter 10 and will be applied to boundary value problems where matrices are now replaced by differential operators. Here there are parallels between symmetric matrices and self-adjoint boundary value problems.

For very large algebraic systems, direct methods such as Gauss elimination require large storage and large computer operations. An alternative is to use iterative methods which are introduced in Chapter 11. The last chapter deals with another problem: not all problems are linear. For example, finding the square root means solve $x^2 - d = 0$. Fortunately, these can be approximated by a sequence of linear solves as in Newton's method to find the square root.

This one-semester course is just the beginning for additional studies. Mathematics courses on numerical analysis, numerical solutions of differential equations, numerical linear algebra and functional analysis might be interesting. Computer science has a number of courses that will help with using particular computer architectures, artificial intelligence and algorithms. Statistical modeling and computing have always been important. Finally, there are many application courses in the science/engineering programs.

The computer codes and updates for this book can be found at the website:

white.math.ncsu.edu/filenames

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